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Approximate Confidence Limits for Complex Systems  
with Exponential Time until Failure of the Components

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APPROXIMATE CONFIDENCE LIMITS FOR COMPLEX SYSTEMS  
WITH EXPONENTIAL TIME UNTIL FAILURE OF THE COMPONENTS

by

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Abstract

The asymptotic distribution of the log-likelihood ratio is shown to provide a method of determining approximate confidence limits for any coherent system when each component has an exponential life with unknown failure rate and component performance data are provided in the form, number of failures (minimum of one) and total operating time. Some computational methods to facilitate the determination of the lower confidence bound at a given level are provided in the important case of a series system. A numerical comparison is made between the nominal confidence level and the actual confidence level, by counting the number of times the true reliability is caught within the confidence bound determined by the likelihood ratio method using repeated computations from random numbers generated with a specified distribution.

## 0. Introduction

The problem of establishing confidence limits for the reliability of systems has now extended over a decade. The first results were those of Buehler [2] in 1957. The problem he considered was equivalent with finding exact confidence limits for two components in series with binomial data on each component. The construction of tables of exact bounds for up to three component series systems with various sample sizes for the components was done by Lipow and Riley [7]. This work in two volumes was published by the Defense Documentation Center.

However, the bulk of the tables for even such small numbers of components made the use of simpler approximate confidence limits quite appealing. Madansky in [9] utilized the asymptotic distribution of the likelihood ratio and the usual practice of inverting a test to obtain a confidence bound, to yield approximate confidence bounds for series, parallel and series-parallel systems.

Recently Lentner and Buehler [6] used the Lehmann-Scheffé theory of exponential families to find exact confidence limits for the specific case of components in series. Due to the difficulty in computing these limits, El Mawaziny and Buehler [4] give an approximation to this exact solution for the case where the sample sizes for all components are large and the failure law for each component is exponential.

El Mawaziny and Buehler make no numerical comparisons of their approximate confidence intervals with those obtained by the exact method, or the approximations using the asymptotic distribution of the maximum likelihood estimates or the likelihood ratio. They do show that under certain conditions on the component samples sizes their confidence

limits are asymptotically equal to both the maximum likelihood limits and to the special case of Bayesian limits where the prior "density" is a particular nonprobabilistic form. They state that they do not believe that their method is applicable to other than series systems.

Approximate confidence intervals for the reliability of any system (or structure) which can be represented by a monotone Boolean function of Bernoulli variates were obtained by Myhre and Saunders [10]. In [10] the component failure data were the outcome of a number of Bernoulli trials for performance or nonperformance. That paper was an extension of the results of Madansky loc. cit., for series systems in that it depended upon the adequacy of the asymptotic distribution of the likelihood ratio. Here we will follow the same general lines of argument used in [10] to obtain approximate confidence intervals for system reliability from samples of component life lengths but the assumption of binomial data on performance of each component is replaced by the assumption of component life length being exponentially distributed.

### 1. The General Coherent System

Let the number of components in a given system be  $m$ . The state of the components, at any given time  $t > 0$ , is the random vector  $\underline{Y}(t) = (Y_1(t), \dots, Y_m(t))$  where  $Y_j(t)$ , a Bernoulli random variable, is the indicator of performance for the  $j^{\text{th}}$  component at that time. It is assumed that the system has a unique representation as a nondecreasing Boolean function  $\phi$ , the functional value of which,  $\phi(\underline{Y}(t))$ , is the indicator of the state of the system. We assume without loss of generality that each component of  $\phi$  is essential, p. 64, [1].

The reliability of the  $j^{\text{th}}$  component is  $EY_j(t)$  and similarly the reliability of the system is  $E\phi(\underline{Y}(t))$  at any time  $t > 0$ .

If the life length of the  $j^{\text{th}}$  component  $X_j = \sup\{t > 0: Y_j(t) = 1\}$  is exponentially distributed for  $j=1, \dots, m$ , then the density of  $X_j$  is

$$(1.1) \quad f_j(t) = \lambda_j \exp(-\lambda_j t) \quad \text{for } t > 0.$$

Let  $\lambda = (\lambda_1, \dots, \lambda_m)$  be a point in the parameter space

$$\mathcal{H} = \{(\lambda_1, \dots, \lambda_m): 0 < \lambda_j < \infty \quad j=1, \dots, m\}.$$

Since we are primarily interested in the system reliability at a prescribed time (in certain cases called the mission length) we can without loss of generality take it to be unity. We now set

$$h(\cdot) = E[Y(1)]$$

$$= \sum_y h(y) \prod_{j=1}^m \exp(-\lambda_j y_j) [1 - \exp(-\lambda_j)]^{1-y_j}$$

where  $y = (y_1, \dots, y_m)$  is a vertex of the  $m$ -dimensional unit hypercube and the summation is over all such vertices. One sees that

$$(1.2) \quad h(\cdot) = \sum_y h(y) \prod_{j=1}^m (y_j \exp(-\lambda_j) + (1-y_j)[1 - \exp(-\lambda_j)]).$$

Suppose that  $n_j \geq 1$  identical replications of the  $j^{\text{th}}$  component are tested for  $j=1, \dots, m$ .

Consequently, observations are made on independent random variables identically distributed as  $X_j$ , which by our convention are life lengths expressed as multiples (possibly less than one) of the given mission length. Call these variables  $X_{ij}$  for  $i=1, \dots, n_j$ .

We postulate that often in practice the random variable which is actually observed is  $\min[X_{ij}, Z_{ij}]$  where  $Z_{ij}$  is the random time (again expressed in terms of mission length) at which the test of the

$i^{th}$  replication of the  $j^{th}$  component will terminate for any reason other than failure of the component. Perhaps  $Z_{ij}$  is degenerate at the fixed time at which the test is terminated, this time being determined by the funds available for testing and known prior to the test's commencement. Alternatively,  $Z_{ij}$  could be the random time of failure of ancillary test equipment which causes the abortion of the test, or any one of a hundred other causes. The important point is that we observe the events

$$(1.3) \quad \begin{aligned} & [X_{ij} = x_{ij}] \cap [X_{ij} < Z_{ij}] \quad i=1, \dots, s_j \\ & [Z_{ij} = z_{ij}] \cap [X_{ij} \geq Z_{ij}] \quad i=s_j+1, \dots, n_j \end{aligned}$$

where  $s_j$  (the number of complete life observations of the  $j^{th}$  component) is an observed value of a random variable determined by the joint distribution of  $Z_{ij} \quad i=1, \dots, n_j$ , which is unspecified. (Under special circumstances the distribution of the number of observed failures is known to be Poisson.)

Sometimes, of course, it is decided prior to the start of the test of the  $j^{th}$  component that it will be terminated at the  $s_j^{th}$  failure. Here  $s_j$  would be a known parameter and the time of completion of the test would become a random variable. The required joint distribution of  $Z_{ij} \quad i=1, \dots, n_j$  can be surmised for this case.

We will now state a well-known result as

Lemma 1: If  $(Z_1, \dots, Z_n)$  is a vector of non-negative random variables independent of  $(X_1, \dots, X_n)$  which are themselves identically distributed non-negative random variables with common density  $f$ , then the likelihood of the event

$$(1.4) \quad \bigcap_{i=1}^k [x_i \leq z_i] [x_i = x_i] \bigcap_{i=k+1}^n [x_i > z_i] [z_i = z_i]$$

is of the form

$$K \prod_{i=1}^k f(x_i) \prod_{j=k+1}^n [1 - F(z_j)]$$

where  $K$  does not depend upon  $f$ .

**Proof.** Let  $G$  be the joint distribution of  $(z_1, \dots, z_n)$ , then the probability of the event specified in (1.4) is

$$\int_{\{z_i > x_i \text{ } i=1, \dots, k\}} \prod_{i=1}^k f(x_i) \prod_{i=k+1}^n [1 - F(z_i)] dG(z_1, \dots, z_n)$$

which upon simplification shows that

$$K = \int_{\{z_i > x_i \text{ } i=1, \dots, k\}} dG(z_1, \dots, z_n),$$

and thus for any observed failure times  $(x_1, \dots, x_k)$ ,  $K$  does not depend upon  $f$ . ||

This lemma, in equivalent forms, was given earlier by Herd [5] and Sampford [11]. Also in this connection see the discussion by Cohen in [2].

Throughout what follows we shall assume that  $z_{ij}$  for  $i=1, \dots, n_j$ , have a distribution such that with probability one

$$(1.5) \quad 0 < s_j \leq n_j.$$

It is clear now that the log-likelihood function for the sample from the  $j^{\text{th}}$  component may be written as

$$\exp\{L_j^*(\lambda)\} = K_j \prod_{i=1}^{s_j} f_j(x_{ij}) \prod_{i=s_j+1}^{n_j} (1 - F_j(z_{ij}))$$

where  $K_j$  is a constant not depending upon the parameter  $\lambda$  of the distribution.

From the specific assumption made in (1.1), by setting  $L^* = \sum_{j=1}^m L_j^*$ , we find

$$(1.5.1) \quad L^*(\lambda) = \sum_{j=1}^m [\ln K_j + s_j \ln \lambda_j - \lambda_j t_j]$$

where for typographical convenience we denote the total test time (in fractions of the mission length) of the  $j^{\text{th}}$  component by

$$(1.6) \quad t_j = \sum_{i=1}^{s_j} x_{ij} + \sum_{i=s_j+1}^{n_j} z_{ij}.$$

The logarithm of the likelihood ratio, say  $L(r)$ , is given by

$$L(r) = \sup_{\{\lambda : h(\lambda) = r\}} L^*(\lambda) - \sup_{\lambda \in \mathcal{H}} L^*(\lambda).$$

We now follow the usual method of inverting a test, in this case the likelihood ratio test, in order to obtain a confidence interval.

Proceeding as we have done in [10] we utilize Wilks' theorem [14] on the asymptotic behavior of the logarithm of the likelihood ratio to

obtain a confidence set of level  $\gamma$  for the system reliability at the mission length. This is

$$(1.6.1) \quad \{r: -2L(r) \leq \chi^2_{\gamma}(1)\}$$

where  $\chi^2_{\gamma}(1)$  is the  $\gamma^{th}$  quantile of the Chi-square distribution with one degree of freedom.

Since the maximum likelihood estimate of  $\lambda_j$  is

$$\hat{\lambda}_j = \frac{s_j}{t_j} \quad j=1, \dots, m$$

we see

$$\begin{aligned} L^*(\cdot) &= \sup_{\lambda \in \mathcal{H}} L^*(\lambda) \\ &= \sum_{j=1}^m [\ln K_j + s_j \ln s_j - s_j (\ln t_j + 1)]. \end{aligned}$$

To maximize  $L^*(\cdot)$  subject to the restriction  $h(\lambda) = r$  we use a Lagrange multiplier  $\delta$ , take partial derivatives, and equate to zero,

$$\frac{\partial}{\partial \lambda} [L^*(\lambda) - \delta h(\lambda)] = 0.$$

This is equivalent with

$$\frac{\partial}{\partial \lambda} \sum_{j=1}^m [s_j \ln \lambda_j - \lambda_j t_j - \delta h(\lambda)] = 0,$$

and thus we obtain

$$(1.7) \quad \frac{s_j}{\lambda_j} - t_j = \delta \partial_j h(\lambda)$$

where  $\partial_j h$  is the partial derivative of  $h$  with respect to  $\lambda_j$ .

The existence of  $\partial_j h$  follows from the definition of  $h$ . For given  $\delta$  denote the vector solution of (1.7) by  $\tilde{\lambda}(\delta)$ , assuming presently that it exists and is unique within  $\mathcal{M}$ , which we shall later prove in Theorem 3. Note that  $\tilde{\lambda}(0) = \hat{\lambda}$ .

Since Lagrange multipliers are being used it will be shown that the confidence set may conveniently be written in terms of the multiplier  $\delta$  rather than in terms of the reliability  $r$ .

First define

$$(1.8) \quad \Lambda(\delta) = L^*[\tilde{\lambda}(\delta)] - L^*(\hat{\lambda})$$

for those values of  $\delta$  for which  $\tilde{\lambda}(\delta)$  exists. It will be shown later in Theorem 4 that the existence of unique  $\tilde{\lambda}_j(\delta)$  implies the existence of the derivative  $\tilde{\lambda}'_j(\delta)$ . Thus it is possible to differentiate  $\Lambda$  with respect to  $\delta$ . Since

$$\Lambda(\delta) = \sum_{j=1}^m [s_j \ln \tilde{\lambda}_j(\delta) - \tilde{\lambda}_j(\delta)t_j - s_j \ln s_j + s_j \ln t_j + s_j]$$

$$\Lambda'(\delta) = \sum_{j=1}^m [\frac{s_j}{\tilde{\lambda}_j(\delta)} - t_j] \tilde{\lambda}'_j(\delta)$$

which by equation (1.7) shows

$$\Lambda'(\delta) = \sum_{j=1}^m \delta \partial_j h[\tilde{\lambda}(\delta)] \tilde{\lambda}'_j(\delta) = \delta \frac{d}{d\delta} h[\tilde{\lambda}(\delta)]$$

From this equation there follows

Lemma 2: At each  $\delta$  for which  $\lambda(\delta)$  exists  $h\lambda(\delta)$  is monotone decreasing in  $\delta$  if and only if  $\Lambda(\delta)$  is monotone decreasing in  $\delta$  for  $\delta > 0$  and monotone increasing in  $\delta$  for  $\delta < 0$ .

Using this result we obtain

Theorem 1: If  $h\lambda(\cdot)$  is monotone decreasing across an interval  $[\delta^-, \delta^+]$  where  $\delta^- < 0 < \delta^+$  are two values of  $\delta$  for which

$$\Lambda(\delta) = -\frac{1}{2} x_\gamma^2(1)$$

then

$$(1.8.1) \quad \{r: h\lambda(\delta^-) + r > h\lambda(\delta^+)\} = \{r: 2L(r) \leq x_\gamma^2(1)\}.$$

Proof. Since  $h\lambda(\delta)$  is a one to one transformation for each given  $r$  for which  $2L(r) \leq x_\gamma^2(1)$ , there exists a unique  $\delta_r$  such that  $h\lambda(\delta_r) = r$ . Thus by definition we have

$$L(r) = L^*[\lambda(\delta_r)] = L^*(\hat{\lambda}) = \Lambda(\delta_r)$$

and the parameterization of  $L$  by  $r$  may be replaced by that of  $\Lambda$  by  $\delta$ . ||

To complete this argument we must show that there exists values of  $\delta$  for which  $\lambda(\delta)$  exists uniquely, and hopefully can be easily found, and also that there exists values of  $\delta$  in an interval about zero for which  $h\lambda(\cdot)$  is decreasing. We turn to the first task now.

The transformation  $A$  from  $\mathcal{H}$  into  $\mathcal{H}$  for fixed  $\delta$  we define by setting its  $j^{\text{th}}$  component for  $j=1, \dots, m$

$$(1.9) \quad A_j(\lambda; \delta) = \frac{s_1}{t_j + \delta \partial_j h(\lambda)}.$$

This is suggested by solving (1.7) as if  $\partial_j h$  were constant. It is clear that  $A_j(\lambda; \delta)$  is continuous in  $\delta$ . Since  $\partial_j h(\lambda) < 0$ , see equation (1.14) below,  $A_j(\lambda; \cdot)$  is an increasing function of  $\delta$  for  $t_j > -\delta \partial_j h(\lambda)$ .

We now quote from [10] a result easily proved, that for  $a, b \in (0, 1)$  and  $\alpha, \beta$  any real numbers

$$(1.10) \quad |a\alpha - b\beta| \leq |\alpha - b| + |a - \beta|.$$

For a given structure  $\phi$  define the criticality of the  $j^{\text{th}}$  component in the structure by

$$(1.11) \quad c_j = \sum_{\substack{y_i \\ i \neq j}} \phi(y|j:1) - \phi(y|j:0)$$

where  $(y|j:x) = (y_1, \dots, y_{j-1}, x, y_{j+1}, \dots, y_m)$  for  $x = 0, 1$ .

We now have

**Theorem 2:** For each  $\phi$  (or  $h$ ) and all  $\delta$  such that

$$(1.12) \quad \min(t_1, \dots, t_m) > \delta > 0 \quad \text{and} \quad \sum_{j=1}^m \frac{s_1 c_j}{(t_j - \delta)^2} < \frac{1}{\delta}$$

$$(1.13) \quad \delta < 0 \quad \text{and} \quad \sum_{j=1}^m \frac{s_j c_j}{t_j^2} < \frac{1}{|\delta|},$$

the transformation  $\Lambda(\cdot; \delta)$  is a contractive map of the complete metric space  $(\mathcal{H}, d)$  into itself where

$$d(\lambda, \mu) = \frac{1}{m} \sum_{j=1}^m |\lambda_j - \mu_j| \quad \text{for } \lambda, \mu \in \mathcal{H}.$$

**Proof.** It must be shown that there exists  $p \in [0, 1)$  such that

$$d[\Lambda(\lambda), \Lambda(\mu)] \leq pd(\lambda, \mu) \quad \text{for all } \lambda, \mu \in \mathcal{H}.$$

Note that throughout this proof we shall omit showing the dependence of both  $\Lambda$  and  $p$  upon  $\delta$  in order to simplify the notation. Now

$$\Lambda_j(\lambda) - \Lambda_j(\mu) = \frac{s_j \delta [\partial_1 h(\mu) - \partial_1 h(\lambda)]}{[t_j + \delta \partial_j h(\lambda)][t_j + \delta \partial_j h(\mu)]}.$$

Since by known properties of coherent systems, see [1]

$$y_j = y_j \psi(y|j:1) + (1-y_j) \phi(y|j:0)$$

substitute  $y_j(1)$  for  $y_j$  and take expectations to obtain

$$h(\lambda) = e^{-\lambda_j} h(\lambda|j:1) + (1-e^{-\lambda_j}) h(\lambda|j:0).$$

Now

$$(1.14) \quad \partial_j h(\lambda) = e^{-\lambda_j} [\partial_1 h(\lambda|j:0) - \partial_1 h(\lambda|j:1)].$$

If we continue the expansion we obtain

$$\partial_j h(\lambda) = e^{-\lambda} \sum_{\substack{y_1 \\ i \neq j}} \{ [h(y|j:0) - h(\lambda|j:1)] \prod_{i \neq j} [y_i e^{-\lambda_i} + (1-y_i)(1-e^{-\lambda_i})] \}.$$

Thus writing out  $\partial_j h(\mu) - \partial_j h(\lambda)$ ; using the definition of (1.11) and applying the inequality (1.10) repeatedly on the last fraction we find:

$$|\partial_j h(\mu) - \partial_j h(\lambda)| \leq c_j \sum_{i=1}^m |e^{-\mu_i} - e^{-\lambda_i}|.$$

$$\text{Since } |e^{-\lambda_i} - e^{-\mu_i}| \leq |\lambda_i - \mu_i|$$

$$|\partial_j h(\mu) - \partial_j h(\lambda)| \leq m c_j d(\lambda, \mu)$$

and

$$d(A(\lambda), A(\mu)) \leq \sum_{j=1}^m \frac{s_j c_j \delta d(\lambda, \mu)}{[t_j + \delta \partial_j h(\lambda)][t_j + \delta \partial_j h(\mu)]}.$$

Since each component is essential it follows from (1.14) that  $\partial_j h(\lambda) < 0$  as  $0 < h(\lambda|j:1) - h(\lambda|j:0) < 1$ . It also follows from (1.14) that  $\partial_j h(\lambda) \geq -e^{-\lambda} \geq -1$ . Thus it is sufficient to require that (1.12) hold for  $\delta > 0$  and (1.13) hold for  $\delta < 0$ . ||

For any  $\lambda^* \in \mathcal{H}$  we define the sequence

$$\lambda^n(\delta) = A(\lambda^{n-1}(\delta); \delta) \quad n=1, 2, \dots$$

where

$$\lambda^*(\delta) = \lambda^*.$$

Theorem 3: For every  $\delta$  in the neighborhood of zero defined by the inequalities (1.12) and (1.13), a unique solution to the system of equations (1.7) exists, call it  $\tilde{\lambda}(\delta)$ . It can be found for any initial point  $\lambda^0 \in \mathcal{H}$  as  $\lim_{n \rightarrow \infty} \lambda^n(\delta) = \tilde{\lambda}(\delta)$ .

Proof. That  $A(\cdot; \delta)$  for  $\delta$  within the prescribed neighborhood of zero has a unique fixed point

$$\tilde{\lambda}(\delta) = \lim_{n \rightarrow \infty} \lambda^n(\delta) = A(\tilde{\lambda}(\delta); \delta)$$

follows from the known behavior of contractive maps, e.g. see p. 27, [8]. ||

Thus we have established the first claim which was made, namely that  $\tilde{\lambda}(\delta)$  exists uniquely. Now we prove

Theorem 4: The function  $\tilde{\lambda}(\cdot)$  is a continuously differentiable function within the neighborhood of zero prescribed by (1.12) and (1.13).

Proof. Fix  $\delta$  within the prescribed neighborhood and let  $B$  be the vector valued function with its  $j^{\text{th}}$  coordinate defined by

$$B_j(\lambda; \delta) = \delta \partial_j h(\lambda) + t_j - \frac{s_j}{\lambda_j}.$$

By Theorem 3 the equation  $B(\lambda; \delta) = 0$  has a unique solution, call it  $\tilde{\lambda}$ . Thus the Jacobian is not zero, i.e.

$$\det \partial_i B_j(\tilde{\lambda}; \delta) \neq 0.$$

From the implicit function theorem the continuous differentiability of the function  $B$  is inherited by  $\tilde{\lambda}$ . Thus if  $\tilde{\lambda}$  exists it is continuously differentiable. ||

In view of Theorems 3 and 4 it remains only to show that  $h\tilde{\lambda}(\delta)$  is a decreasing function of  $\delta$ . We then have the result that for any coherent structure  $\phi$  an approximate confidence interval for the reliability function  $h$  is given by (1.8.1) in Theorem 1.

Theorem 5: For any coherent structure  $\phi$  with reliability function  $h$  and any failure data such that  $s_j \geq 1$ , there exists a neighborhood of zero in  $\delta$  across which  $h\tilde{\lambda}(\delta)$  is decreasing in  $\delta$ .

Proof. Take the derivative of both sides of (1.7) with respect to  $\delta$ , primes denoting such differentiation, we obtain for  $j=1, \dots, m$

$$(1.15) \quad \frac{-s_j}{(\tilde{\lambda}_j)^2} \cdot \tilde{\lambda}'_j = \partial_j h\tilde{\lambda} + \delta \sum_{i=1}^m \partial_{ij} h(\tilde{\lambda}) \cdot \tilde{\lambda}'_i$$

where we have omitted the argument  $\delta$ . Multiply by  $\tilde{\lambda}'_j$  on both sides of (1.15) and sum to obtain

$$-\sum_{j=1}^m \frac{s_j}{(\tilde{\lambda}_j)^2} (\tilde{\lambda}'_j)^2 = \sum_{j=1}^m \partial_j h(\tilde{\lambda}) \cdot \tilde{\lambda}'_j + \delta \sum_{i=1}^m \sum_{j=1}^m \partial_{ij} h(\tilde{\lambda}) \tilde{\lambda}'_i \tilde{\lambda}'_j$$

which is equivalent with,

$$(1.16) \quad -\frac{d}{d\delta} h[\tilde{\lambda}(\delta)] = \sum_{j=1}^m \frac{s_j}{(\tilde{\lambda}_j)^2} (\tilde{\lambda}'_j)^2 + \delta \sum_{i=1}^m \sum_{j=1}^m \partial_{ij} h(\tilde{\lambda}) \tilde{\lambda}'_i \tilde{\lambda}'_j.$$

For  $\delta = 0$ , (1.16) is positive and from the continuity of  $\frac{d}{d\delta} h[\tilde{\lambda}(\delta)]$  it follows that there exists a neighborhood of  $\delta$  about zero such that  $h\tilde{\lambda}(\delta)$  is a decreasing function of  $\delta$ . ||

Thus for any coherent structure where the data on each component's reliability consists of knowing the times of failure or the lengths of time a component worked without failure, assuming the life length of the individual components are exponentially distributed, it is possible to find an approximate confidence interval of some level about the maximum likelihood estimate of the system reliability at the given (mission) time. Of course, it may not be possible that an approximate confidence interval of arbitrary level can be found for every structure and every set of sample data, but it is always possible to obtain an interval estimate.

## 2. The Series System

For any coherent structure  $\phi$  with associated reliability function  $h$ , there exists a series system, the reliability of which bounds the system reliability  $h$  from below, i.e.

$$(2.1) \quad h(\lambda) \geq \exp \left\{ - \sum_{j=1}^m v_j \lambda_j \right\}$$

where  $v_j$  is the number of minimal cuts of  $h$  (or  $\phi$ ) which contain  $j$  as a member. For the exact definition of minimal cut and a more detailed discussion of this point, see [12].

Because of this fact we will, in this section consider only the case  $\ln h(\lambda) = -\sum v_j \lambda_j$  and obtain a lower confidence bound. We think this is the most important case.

We will follow the general procedure which has been presented in the preceding section. However, as a convenience we reparametrize by imposing the restriction in the form that the  $\ln h(\lambda)$  is constant, and equate

$$\frac{\partial}{\partial \lambda_j} [L^*(\lambda) - \delta \ln h(\lambda)] = 0.$$

Thus we obtain, instead of (1.7), the equation

$$(2.2) \quad \frac{s_1}{\lambda_j} - t_j = -\delta v_j$$

which has the explicit solution

$$\tilde{\lambda}_j(\delta) = \frac{s_1}{t_j - \delta v_j} \quad j=1, \dots, m.$$

Thus in this particular case, it is unnecessary to use the general contractive map theorems in order to calculate  $\tilde{\lambda}(\delta) = (\tilde{\lambda}_1(\delta), \dots, \tilde{\lambda}_m(\delta))$ . For this special case we find, using (1.5.1) and (1.8),

$$\Lambda(\delta) = \sum_{j=1}^m s_j [1 - \frac{\tau_j}{\tau_j - \delta} + \ln(\frac{\tau_j}{\tau_j - \delta})]$$

where  $\tau_j = t_j/v_j$  for  $j=1, \dots, m$ . In accord with finding only a lower confidence bound, we want to solve for  $x > 0$  such that

$$(2.3) \quad \Lambda(x) = -\frac{1}{2} \chi_{\gamma}^2(1),$$

where  $\chi_{\gamma}^2(1)$  is the  $100\gamma^{th}$  percentile of the Chi-square distribution with one degree of freedom. Call the solution of (2.3) the value  $\delta_{\gamma}^+$ .

Then

$$h[\tilde{\lambda}(\delta_{\gamma}^+)] = \exp \left\{ - \sum_{j=1}^m \frac{s_j}{\tau_j - \delta_{\gamma}^+} \right\}$$

is an (approximate) lower bound for  $h(\lambda)$  of level  $\frac{1+\gamma}{2}$ , rather than of level  $\gamma$ , since we are obtaining only a one-sided confidence bound.

We now exhibit a practical method for the determination of  $\delta_{\gamma}^+$ .

For a given  $\gamma$ ,  $0 < \gamma < 1$  there exists an  $x > 0$  such that  $f(x) = 0$  where

$$(2.4) \quad f(x) = -\Lambda(x) - \frac{1}{2} \chi_{\gamma}^2(1) \\ = \sum_{i=1}^m s_i [\frac{\tau_i}{\tau_i - x} - \ln(\frac{\tau_i}{\tau_i - x})] - \sum_{i=1}^m s_i - \frac{1}{2} \chi_{\gamma}^2(1).$$

The solution of  $f(x) = 0$  must be unique since

$$f'(x) = x \sum_{i=1}^m \frac{s_i}{(\tau_i - x)^2} > 0$$

whenever  $0 < x < \min(\tau_1, \dots, \tau_m) = \tau_{(1)}$ , which is a more stringent condition than (1.12). But note that

$$f''(x) = \sum_{i=1}^m s_i \frac{\tau_i + x}{(\tau_i - x)^3} > 0$$

for  $0 < x < \tau_{(1)}$ .

Thus we see that both  $f', f''$ , which are continuous, do not vanish for  $0 < x < \tau_{(1)}$ . These are sufficient conditions that the Newton iteration procedure, namely

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \quad n=1,2,\dots,$$

will converge to the value  $\delta_\gamma^+$ .

We now state the

**Theorem 6:** Let the data  $(t_i, s_i)$   $i=1,\dots,m$  be given, where  $t_i$  represents the total test time for the  $i^{th}$  component and  $s_i \geq 1$ , represents the number of failures of that component during that time. If  $v_i$  is the criticality of the  $i^{th}$  component in any structure being measured by the number of minimal cuts of the system which contain the  $i^{th}$  component as a member, and  $\tau_i = t_i/v_i$ , then

$$(2.5) \quad \begin{aligned} & \exp \left\{ - \sum_{j=1}^m \frac{s_j}{(\tau_j/t) - \delta_Y^+} \right\} \quad \text{for } 0 < t < \tau_{(1)} / \delta_Y^+ \\ & 0 \quad \text{for } t \geq \tau_{(1)} / \delta_Y^+ \end{aligned}$$

is a lower confidence bound of level  $\frac{1+\gamma}{2}$ , where  $0 < \gamma < 1$  is preselected, for the system reliability for all time  $t > 0$ . The quantity  $\delta_Y^+$  is the solution of the equation  $f(x) = 0$  and  $f$  was defined in equation (2.4).

Proof. We have only to remark that the selection of a time scale in terms of mission length was arbitrary so that the bound derived previously can be used for the reliability at all times. ||

### 3. A Numerical Illustration

As a numerical illustration of the behavior of the previously outlined procedure we shall study its performance on a specified series system with ten components. Take  $m = 10$ . We shall assume that  $\lambda_j = 1$  for  $j=1, \dots, 10$ . Thus the true reliability of the system is, by (1.2)  $e^{-10t}$  for  $t > 0$ . We also assume that  $n_j = v_j = 1$  and thus  $\tau_j = t_j$  for  $j=1, \dots, m$ .

We will now generate 10 exponential variates,  $t_1, \dots, t_m$ , each with unit mean and solve the equation  $f(x) = 0$  where

$$(3.1) \quad f(x) = \sum_{i=1}^m \left[ \frac{t_i}{t_i - x} - \ln\left(\frac{t_i}{t_i - x}\right) \right] - c_\gamma$$

and  $c_\gamma$  is a constant determined by the nominal level of the confidence bound.

We find that for  $\gamma = .95$ ,  $c_{.95} = 11.353$ . Call the solution of (3.1) the value  $\delta_\gamma^+$ . The lower bound for the reliability is by (2.5)

$$(3.2) \quad \exp \left\{ - \sum_{i=1}^m \left( \frac{1}{t_i / t - \delta_\gamma^+} \right) \right\}, \quad \text{for } 0 < t < \delta_\gamma^+ t_{(1)},$$

and the confidence level is 97.5.

Generating forty independent observations of an exponential variate with unit mean, resulted in the following four samples:

(1)	(2)	(3)	(4)
1.305	1.889	.2331	3.258
1.346	3.864	.8881	.1848
1.483	.7789	.04059	.7594
.2931	.4825	.8329	2.420
.2282	1.171	.1948	.8638
.3489	.5642	1.332	.5307
1.342	.3525	2.105	.06695
.7171	.9037	1.144	.8239
.3397	.5590	1.203	.2452
1.126	.5445	.07427	.2306

Solving (3.1) by machine program and calculating (3.2) for each set of observations yielded the confidence bounds which are summarized visually in Figure 1.

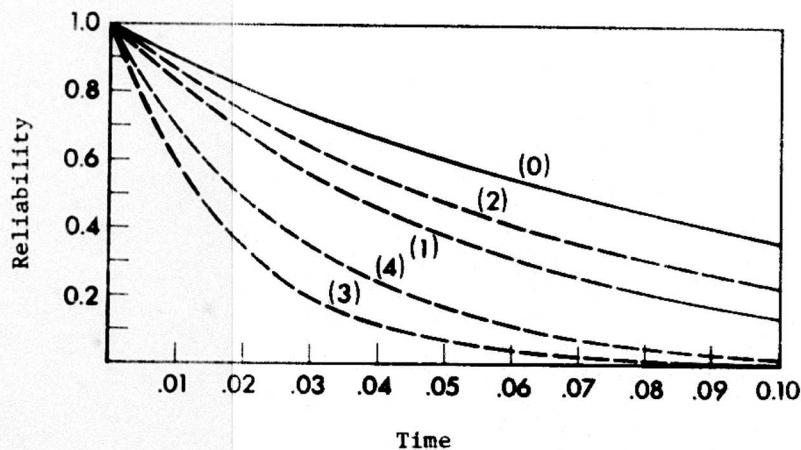


Figure 1. 97.5% lower confidence bounds on the reliability of a 10 component system based on independent samples.

(0) is the true reliability  $e^{-10t}$  for  $t > 0$

(i) is the confidence bound based on sample (i) for  $i=1,2,3,4$ .

As a further illustration of the stochastic behavior of this method we shall repeat the entire procedure two thousand times but instead of calculating the entire lower bound function we shall only compute (3.2) at  $t = .01$ . Then we shall make a frequency histogram of the values of the reliability bounds for the specified value  $t = .01$ . The twenty percentile points, successive differences being .05, of the empiric distribution are:

Percentile	Percentile Point
.05	0.106621
.10	0.297174
.15	0.417770
.20	0.492659
.25	0.552798
.30	0.596595
.35	0.632835
.40	0.665383
.45	0.689521
.50	0.707729
.55	0.730936
.60	0.750021
.65	0.767430
.70	0.782541
.75	0.798792
.80	0.815686
.85	0.833291
.90	0.851064
.95	0.877483
1.00	0.940218

A graph of the empiric distribution is given in Figure 2. Note that the true reliability of  $e^{-1} = .904837$  slightly exceeded the nominal 97.5 percentile. In fact the actual count was 1979 values less than  $e^{-1}$  out of the 2000 observations sampled. This was approximately the 99th percentile.

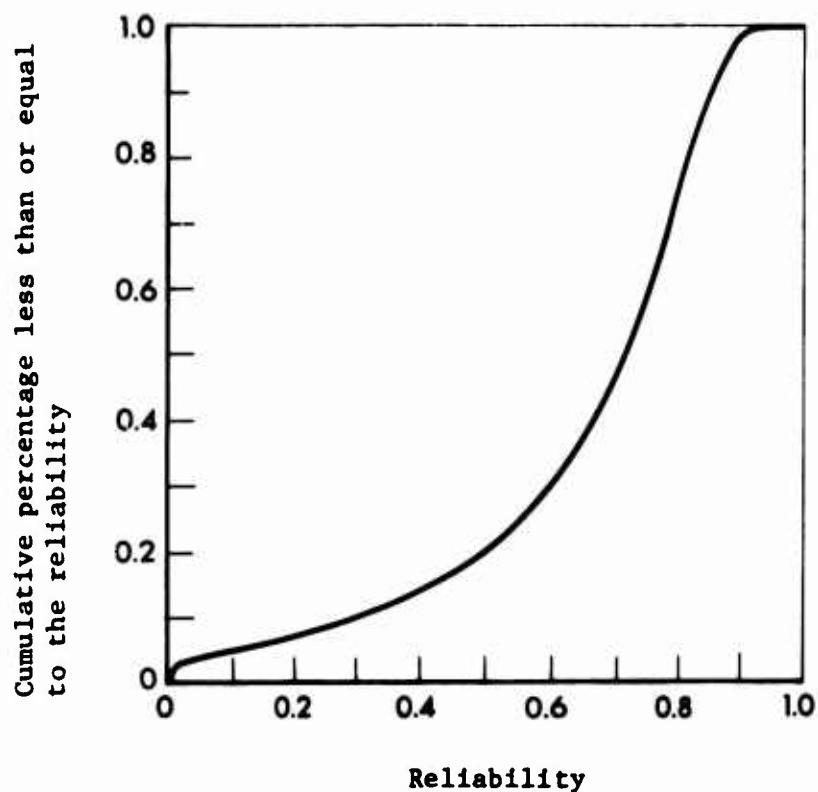


Figure 2. Empiric distribution of the 97.5 per cent lower confidence bound using 2000 observations when the true reliability is  $e^{-1} = .905$ .

Thus we see the stated confidence level of 97.5 percent appears to be, in this instance, slightly conservative. To check this we repeated this entire experiment a second time and we observed 1990 out of 2000 observations of the reliability bound were less than  $e^{-1}$  which increases our suspicion of a slightly conservative tendency for the level of confidence in this case.

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